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Fluctuation–dissipation relations in trap models

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Abstract

Trap models are intuitively appealing and often solvable models of glassy dynamics. In particular, they have been used to study ageing and the resulting out-of-equilibrium fluctuation–dissipation relations between correlation and response functions. In this paper I show briefly that one such relation, first given by Bouchaud and Dean, is valid for a general class of mean-field trap models: it relies only on the way a perturbation affects the transition rates, but is independent of the distribution of trap depths and the form of the unperturbed transition rates, and holds for all observables that are uncorrelated with the energy. The model with Glauber dynamics and an exponential distribution of trap depths, as considered by Barrat and Mézard, does not fall into this class if the perturbation is introduced in the natural way by shifting all trap energies. I show that a similar relation between response and correlation nevertheless holds for the out-of-equilibrium dynamics at low temperatures. The results point to intriguing parallels between trap models with energetic and entropic barriers.

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1. Introduction

Trap models consist of a single particle, or equivalently an ensemble of non-interacting particles, hopping in a landscape of traps of energy E . Such models have been studied extensively and shown to account qualitatively for many interesting features of glassy dynamics, see, e.g., [1–10]. In the simplest case the rates for transitions from one trap to another depend only on the energies of the two traps. One then has a mean-field trap model, where no information on any spatial organization is retained. This is the case that will concern us here; for work on spatial trap models, see, e.g., [11–14].

In this paper I focus on the behaviour of trap models after a quench into the glassy phase, and in particular on two-time correlation and response functions. For a generic observable m the correlation function is $C(t, t_w) = \langle m(t)m(t_w) \rangle - \langle m(t) \rangle \langle m(t_w) \rangle$, while the (linear)

response function $\chi(t, t_w)$ measures the change in $\langle m(t) \rangle$ due to a conjugate field h that is switched on at time $t_w < t$. The time of preparation of the system by quenching is taken as the zero of the time axis, so that the waiting time t_w can alternatively be thought of as the ‘age’ of the system at the time when the field is applied. Over the last decade it has been recognized that out-of-equilibrium fluctuation–dissipation (FD) relations between such correlation and response functions are extremely useful for characterizing glassy dynamics [15–18]. One defines a fluctuation–dissipation theorem (FDT) ‘violation factor’ $X(t, t_w)$ by

$$-\frac{\partial}{\partial t_w} \chi(t, t_w) = \frac{X(t, t_w)}{T} \frac{\partial}{\partial t_w} C(t, t_w) \quad (1)$$

so that $X = 1$ corresponds to the usual equilibrium FDT. The value of X can be read off from the slope of a parametric ‘FD plot’ of χ versus C ; see [9, 19] for a discussion of the effects of using either t or t_w as curve parameter. The quantity on the lhs of (1), denoted $R(t, t_w)$ below, is the response to a short field impulse at time t_w .

In glassy systems one typically finds that the decay of two-time correlation functions $C(t, t_w)$ exhibits several regimes: an initial decay to a plateau, with further relaxation taking place only on ‘ageing’ timescales $t - t_w$ that grow with the age t_w , for example $t - t_w = \mathcal{O}(t_w)$. In mean-field spin glasses [15–17] one finds that, in the limit of large t_w , X has well-defined and distinct values in these regimes, corresponding to an FD plot made up of two straight line segments: $X = 1$ in the short-time regime, corresponding to quasi-equilibrium, and $X < 1$ in the ageing regime. In the latter, one can then define an *effective temperature* by $T_{\text{eff}} = T/X$. This has been shown to have many of the properties of a thermodynamic temperature [15–17], opening up the exciting prospect of an effective equilibrium description of out-of-equilibrium dynamics.

The existence of effective temperatures in systems other than the now canonical (e.g. spherical p -) spin glass models has been the subject of much research in recent years, but a coherent picture has yet to emerge [18]. In Bouchaud’s trap model [1], intriguing results have recently been found [9]: even though the correlation functions $C(t, t_w)$ decay within a single ageing ‘time sector’ $t - t_w = \mathcal{O}(t_w)$, the FDT violation factor X is not constant as one might expect by analogy with mean-field spin glass models. Instead it varies continuously with $(t - t_w)/t_w$, resulting in a curved FD plot with an asymptotic slope (for $(t - t_w)/t_w \rightarrow \infty$, i.e. $C \rightarrow 0$) of $X_\infty = 0$.

In the Bouchaud trap model glassy dynamics arises from the presence of *energy* barriers. The aim of this paper is to analyse the FD relations in a different trap model, due to Barrat and Mézard [2], where glassiness instead results from the presence of *entropic* barriers. Because of the different physical mechanisms causing the out-of-equilibrium behaviour, it is then not *a priori* clear whether the FD relations of the two models should be related. I find that some important aspects of the FD relations are indeed the same, pointing to intriguing parallels between models with energetic and entropic barriers that deserve to be explored further.

Very recently, Ritort [10] has also considered FD relations in the Barrat and Mézard model. However, he assumed that the effect of the perturbing field on the transition rates defining the trap model dynamics has a simple multiplicative form. This is ‘not easy to justify *a priori*’ [10] and gives only an approximation to the natural prescription where the effect of the perturbing field is to shift all energies according to $E \rightarrow E - hm$. I show in this paper that the response in this natural Barrat and Mézard model can be analysed directly, and I give exact results for the FD relations in the limit of low temperatures; these differ in important respects from those obtained with the approximation of multiplicatively perturbed rates. As a by-product of the calculation, I also show briefly that the FD relation due to Bouchaud and Dean [4], which was recovered by Ritort [10] for the Barrat and Mézard model with the

approximation of multiplicatively perturbed rates, is in fact valid for *arbitrary* trap models with multiplicatively perturbed rates.

In the following section I give the definitions of the Bouchaud and the Barrat and Mézard trap models; general expressions for correlation and response functions that apply to all trap models are then derived in section 3. In section 4 we apply these to the case of multiplicatively perturbed rates. Section 5 contains the main result, the exact low- T FD relation for the Barrat and Mézard model. I conclude in section 6 with a discussion of the intriguing links between trap models with energetic and entropic barriers which arise as a consequence of this relation.

2. Trap models

A trap model is defined by a distribution $\rho(E)$ of trap energies; the convention for the sign of E here is that lower E corresponds to *deeper* traps, which is the reverse of that in, e.g., [4]. The primary dynamical quantity is then $P_0(E, t)$, the distribution of finding the particle in a trap of energy E at time t ; the subscript 0 indicates that for now we are considering the dynamics without any perturbing fields. The evolution of P_0 is given by the master equation

$$\frac{\partial}{\partial t} P_0(E, t) = -\Gamma_0(E) P_0(E, t) + \rho(E) \int dE' w_0(E \leftarrow E') P_0(E', t) \quad (2)$$

where $w_0(E \leftarrow E')$ is the rate for transitions between traps of energy E' and E . More precisely, if one considers a finite number of traps N , the transition rate from trap i to j is $(1/N)w_0(E_j \leftarrow E_i)$; the total rate for transitions to traps in the energy range $E < E_j < E + dE$ is then $w_0(E \leftarrow E_i)$ times the fraction of traps in this range, which is $\rho(E) dE$ for large N . The quantity

$$\Gamma_0(E) = \int dE' \rho(E') w_0(E' \leftarrow E) \quad (3)$$

in (2) is the total ‘exit rate’ from a trap of energy E .

Two specific instances of trap models have received considerable attention in recent years. Bouchaud [1] chose for his trap model $\rho(E) = T_g^{-1} \exp(E/T_g)$ with $-\infty < E < 0$. For any choice of transition rates that satisfies detailed balance, the model then has a glass transition at $T = T_g$ since the equilibrium distribution $P_{\text{eq}}(E) \propto \rho(E) \exp(-E/T)$ becomes unnormalizable there. For lower T , the system must show ageing, i.e. a strong dependence of its properties on the waiting time t_w . Bouchaud [1] assumed transition rates

$$w_0(E' \leftarrow E) = \exp(\beta E) \quad (4)$$

that are independent of the energy of the arrival trap; here $\beta = 1/T$ as usual. Barrat and Mézard [2] chose instead Glauber rates

$$w_0(E' \leftarrow E) = \frac{1}{1 + \exp[\beta(E' - E)]}. \quad (5)$$

As emphasized by Ritort [10], the out-of-equilibrium dynamics of these models is rather different: in the Bouchaud model with its activated dynamics, glassiness arises from the presence of *energy* barriers, and the system arrests completely for $T \rightarrow 0$. In the Barrat and Mézard case, on the other hand, the system can keep evolving by transitions to traps with ever lower energies, even at $T = 0$; the diminishing number of such traps effectively creates *entropic* barriers that slow the relaxation.

3. Correlation and response

We now want to consider the correlation and response properties of some, essentially arbitrary, observable m . In the most general terms its properties are described by the distributions $\rho(m|E)$ of m across traps of given E . I will assume throughout that m is on average uncorrelated with E , so that its conditional mean

$$0 = \int dm m \rho(m|E) \quad (6)$$

vanishes for all E ; the variance

$$\Delta^2(E) = \int dm m^2 \rho(m|E) \quad (7)$$

however can be dependent on E . With m included, the master equation is

$$\begin{aligned} \frac{\partial}{\partial t} P(E, m, t) &= -\Gamma(E, m) P(E, m, t) \\ &+ \rho(m|E) \rho(E) \int dE' dm' w(E, m \leftarrow E', m') P(E', m', t) \end{aligned} \quad (8)$$

where the rates $w(E, m \leftarrow E', m')$ may now depend on a perturbing field h conjugate to m , and the total exit rates are

$$\Gamma(E, m) = \int dE' dm' \rho(m'|E') \rho(E') w(E', m' \leftarrow E, m). \quad (9)$$

An expression for the *correlation function* of m is easily found. In the absence of a field, $w(E', m' \leftarrow E, m) = w_0(E' \leftarrow E)$ and $\Gamma(E, m) = \Gamma_0(E)$ are independent of the value of our observable. Equation (8) then shows that $P(E, m, t) = \rho(m|E) P_0(E, t)$ as long as the same is true at time $t = 0$. (This is a natural assumption and holds, e.g., when $P(E, m, 0)$ is an equilibrium distribution at zero field and some initial temperature above T_g , from which the system is quenched to $T < T_g$ at $t = 0$.) For our zero-mean observables (6) this implies in particular that $\langle m(t) \rangle = 0$ at all times. The two-time correlator of m is then

$$C(t, t_w) = \langle m(t) m(t_w) \rangle \quad (10)$$

$$= \int dE dm dE' dm' m m' P_0(E, m|E', m', t - t_w) \rho(m'|E') P_0(E', t_w). \quad (11)$$

Here $P_0(E, m|E', m', t - t_w)$ is the propagator, i.e. the probability of being in a trap with energy E and observable m when starting from a trap with E' and m' a time $t - t_w$ earlier. This can be obtained as the solution to (8) starting from the initial condition $\delta(E - E') \delta(m - m')$. Since the correlation function is calculated in the absence of a field, the only nontrivial m -dependence in (8) arises from the factor $\rho(m|E)$. Treating the second term on the rhs of (8) as an inhomogeneity one thus sees that

$$P_0(E, m|E', m', t - t_w) = e^{-\Gamma_0(E')(t-t_w)} \delta(E - E') \delta(m - m') + \rho(m|E) \times \dots \quad (12)$$

where the dots indicate factors not involving m . Inserting into (11) and using the zero-mean assumption (6) then yields the simple representation

$$C(t, t_w) = \int dE \Delta^2(E) e^{-\Gamma_0(E)(t-t_w)} P_0(E, t_w) \quad (13)$$

for the correlation function. This makes sense: physically, every hop completely decorrelates the observable, so that C is an average of the probabilities $\exp[-\Gamma_0(E)(t - t_w)]$ of remaining in the current trap, weighted by the probability of being in a trap of energy E at time t_w and multiplied by the variance of m across traps of this energy.

To find the *impulse response* $R(t, t_w)$, consider a field impulse of amplitude h and infinitesimal length Δt , applied at time t_w . Denote

$$\Delta w(E', m' \leftarrow E, m) = w(E', m' \leftarrow E, m) - w_0(E' \leftarrow E) \quad (14)$$

the change in the transition rates caused by the field, and $\Delta\Gamma(E, m)$ similarly the change in the total exit rates; h -dependences are not written explicitly here. Then from the master equation (8), and using that $P(E, m, t_w) = \rho(m|E)P_0(E, t_w)$, one has

$$\begin{aligned} P(E, m, t_w + \Delta t) &= \rho(m|E)P_0(E, t_w) - \Delta t \Gamma_0(E)\rho(m|E)P_0(E, t_w) \\ &+ \Delta t \rho(m|E)\rho(E) \int dE' dm' w_0(E \leftarrow E')\rho(m'|E')P_0(E', t_w) \\ &- \Delta t \Delta\Gamma(E, m)\rho(m|E)P_0(E, t_w) + \Delta t \rho(m|E)\rho(E) \\ &\times \int dE' dm' \Delta w(E, m \leftarrow E', m')\rho(m'|E')P_0(E', t_w) \end{aligned} \quad (15)$$

where the effects of the field have been explicitly separated off in the last two lines. After time $t_w + \Delta t$, when the field is switched off again, the same argument that leads to (12) applies and so

$$P(E, m, t) = e^{-\Gamma_0(E)(t-t_w)} P(E, m, t_w + \Delta t) + \rho(m|E) \times \dots \quad (16)$$

for $t > t_w + \Delta t$ with the dots again indicating factors independent of m ; in the exponent I have approximated $t - t_w - \Delta t \approx t - t_w$ since we are interested in the limit $\Delta t \rightarrow 0$. To find $\langle m(t) \rangle$, from which the response function is obtained, one inserts (15) into (16), multiplies by m and integrates over m and E . All terms of the form $\rho(m|E) \times \dots$ give a vanishing contribution due to (6). Only the last two lines of (15) thus survive, and the two-time response function can be written as

$$\begin{aligned} hR(t, t_w) &= \frac{1}{\Delta t} \langle m(t) \rangle \quad (17) \\ &= \int dE dm e^{-\Gamma_0(E)(t-t_w)} m \left[-\Delta\Gamma(E, m)\rho(m|E)P_0(E, t_w) \right. \\ &\quad \left. + \rho(m|E)\rho(E) \int dE' dm' \Delta w(E, m \leftarrow E', m')\rho(m'|E')P_0(E', t_w) \right]. \end{aligned} \quad (18)$$

So far this applies for arbitrary field amplitude h , so that $R(t, t_w)$ is in general a nonlinear response function, but we will specialize to the linear response limit $h \rightarrow 0$ below.

4. Multiplicatively perturbed rates

To get concrete expressions for the response function one needs to define how the field h affects the transition rates. The natural prescription is that all energies are shifted according to the value of the observable, $E \rightarrow E - hm$ and $E' \rightarrow E' - hm'$. Before going on to consider the more complicated case of the Barrat and Mézard model, I first briefly review the situation in the Bouchaud model, where a simple relation between correlation and response exists [4]. The derivation will show that this relation actually applies rather generally, being dependent only on the way the field affects the transition rates.

For the Bouchaud model, shifting the energy $E \rightarrow E - hm$ in (4) gives the transition rate in the presence of a field $w(E', m' \leftarrow E, m) = \exp(\beta E - \beta hm) = \exp(-\beta hm)w_0(E' \leftarrow E)$. More generally, one can consider rates perturbed by the field according to [4]

$$w(E', m' \leftarrow E, m) = e^{\beta h[(1-\xi)m' - \zeta m]} w_0(E' \leftarrow E) \quad (19)$$

which reduces to the natural choice¹ for $\zeta = 1$ but also maintains detailed balance for other values of ζ . To linear order in h one then has

$$\Delta w(E', m' \leftarrow E, m) = \beta h [(1 - \zeta)m' - \zeta m] w_0(E' \leftarrow E) \quad (20)$$

and the corresponding change in the exit rates (9) is

$$\Delta \Gamma(E, m) = \beta h \int dE' dm' [(1 - \zeta)m' - \zeta m] \rho(m'|E') \rho(E') w_0(E' \leftarrow E). \quad (21)$$

Using again the zero-mean assumption (6), the first term in square brackets vanishes, giving with (3)

$$\Delta \Gamma(E, m) = -\beta h \zeta m \Gamma_0(E). \quad (22)$$

One can now substitute (20)—with the arguments (E, m) and (E', m') interchanged appropriately—and (22) into (18). Using $\int dm' m' \rho(m'|E') = 0$ and dividing by h yields for the linear response function

$$R(t, t_w) = \beta \int dE \Delta^2(E) e^{-\Gamma_0(E)(t-t_w)} \left[\zeta \Gamma_0(E) P_0(E, t_w) + (1 - \zeta) \rho(E) \int dE' w_0(E \leftarrow E') P_0(E', t_w) \right]. \quad (23)$$

From (13), the first term in square brackets is seen to give $-\beta \zeta \partial C(t, t_w) / \partial t$. For the second one, one notes from (13) and (2) that

$$\frac{\partial C}{\partial t_w} = \int dE \Delta^2(E) e^{-\Gamma_0(E)(t-t_w)} \left[\Gamma_0(E) P_0(E, t_w) + \frac{\partial P_0}{\partial t_w}(E, t_w) \right] \quad (24)$$

$$= \int dE \Delta^2(E) e^{-\Gamma_0(E)(t-t_w)} \rho(E) \int dE' w_0(E \leftarrow E') P_0(E', t_w) \quad (25)$$

which apart from prefactors is just the second term in (23). Thus, for *any* mean-field trap model with the multiplicatively perturbed rates (19), and any zero-mean observable, one has the result given by Bouchaud and Dean [4] for Bouchaud's trap model

$$R(t, t_w) = -\beta \zeta \frac{\partial C}{\partial t} + \beta (1 - \zeta) \frac{\partial C}{\partial t_w}. \quad (26)$$

The above calculation shows that this relation holds entirely independently of the precise form of the trap depth distribution² $\rho(E)$ or the transition rates $w_0(E' \leftarrow E)$. In equilibrium, where $C(t, t_w)$ is a function of $t - t_w$ only, it of course recovers the usual FDT, $R(t, t_w) = \beta \partial C(t, t_w) / \partial t_w$. Equation (26) applies in particular to (zero-mean) *neutral* observables [9, 18], where m is completely decoupled from E and therefore $\rho(m|E)$ is independent of E . It remains true also for more general observables, however, as long as they have zero conditional mean (6).

¹ The multiplicative perturbation of rates (19) arises from the natural energy shift prescription $E \rightarrow E - hm$ only for the activated rates (4). However, it has been advocated also as an approximate treatment for, e.g., Glauber rates (5) [10], and so is worth considering for general $w_0(E' \leftarrow E)$.

² The irrelevance of the form of $\rho(E)$ may well have been known to the authors of [4], but was not stated there. The version of (26) given in [4] is nevertheless more limited than that given here, since only activated rates (4) and neutral observables m were considered.

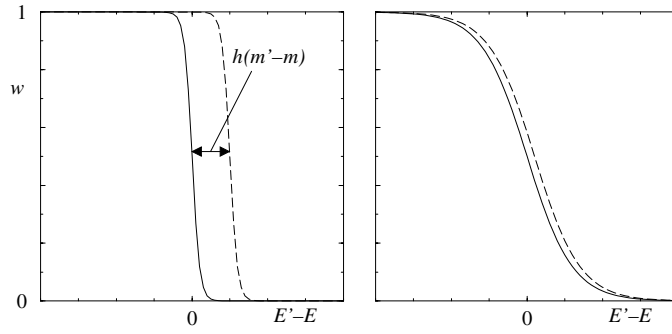


Figure 1. Effect of a field on Glauber transition rates, sketched for $h(m' - m) > 0$. Solid lines show the original transition rates (5), dashed lines those in the presence of a field h , which are shifted to the right by $h(m' - m)$; see arrow on the left. The difference between the two curves is the change in the rates, equation (30); its integral is clearly $h(m' - m)$. Left: case where $|h(m' - m)| > T$; the range where the change is significant is given by $h(m' - m)$. Right: case where $|h(m' - m)| < T$; here the temperature T sets the range where rates change significantly. For small h and small T the range is small in either case.

5. The Barrat and Mézard model

Next let us turn to the Barrat and Mézard model, with the natural prescription which assumes that the field shifts all energies. From (5) the rates are then

$$w(E', m' \leftarrow E, m) = \frac{1}{1 + \exp\{\beta[(E' - hm') - (E - hm)]\}}. \quad (27)$$

For low T , Ritort [10] argued that as a reasonable approximation to this one could consider multiplicatively perturbed rates

$$w(E', m' \leftarrow E, m) = e^{h(\gamma m' - \mu m)} w_0(E' \leftarrow E) \quad (28)$$

with $\gamma = \mu = 1/T_g$ for exponential $\rho(E)$. Equation (28) is identical to (19) apart from the replacements $\beta(1 - \zeta) \rightarrow \gamma$, $\beta\zeta \rightarrow \mu$. As expected from the general result (26) for multiplicatively perturbed rates, Ritort therefore obtained the relation

$$R(t, t_w) = -\mu \frac{\partial C}{\partial t} + \gamma \frac{\partial C}{\partial t_w} \quad (29)$$

between response and correlation. This was found confirmed in simulations. However, as discussed in the appendix, these simulations were effectively performed directly with the approximate rates (28), so did not give a check of how well this approximation captures the behaviour of the Barrat and Mézard model. I now show that the response can be calculated exactly even with the exact rates (27), and that the results differ from those found for multiplicatively perturbed rates.

To calculate the response function, consider the change in the transition rates due to the field,

$$\Delta w(E', m' \leftarrow E, m) = \frac{1}{1 + \exp\{\beta[(E' - hm') - (E - hm)]\}} - \frac{1}{1 + \exp[\beta(E' - E)]}. \quad (30)$$

This is significantly different from zero only for E' within a range of order $\max\{T, |h(m' - m)|\}$ around E ; see figure 1. If this range is small compared to T_g , which is true for $T \ll T_g$ and small fields h , then in

$$\Delta \Gamma(E, m) = \int dE' dm' \rho(m'|E') \rho(E') \Delta w(E', m' \leftarrow E, m) \quad (31)$$

we can to leading order replace E' by E in the factor $\rho(E')$; the same is true for the first factor if we assume that $\rho(m'|E')$ varies with E' at most on the same scale ($\sim T_g$) as $\rho(E')$. Using $\int dE' \Delta w(E', m' \leftarrow E, m) = h(m' - m)$, which from figure 1 is geometrically obvious, together with (6) one thus finds

$$\Delta\Gamma(E, m) = \int dm' h(m' - m) \rho(m'|E) \rho(E) = -hm\rho(E). \quad (32)$$

The same argument can be applied to the integral in the second term of (18), as long as we are in an out-of-equilibrium regime where $P_0(E', t_w)$ varies with E' on a scale of T_g , rather than T as it would in equilibrium. This gives to leading order

$$\begin{aligned} & \int dE' dm' \Delta w(E, m \leftarrow E', m') \rho(m'|E') P_0(E', t_w) \\ &= \int dm' h(m - m') \rho(m'|E) P_0(E, t_w) = hm P_0(E, t_w). \end{aligned} \quad (33)$$

One can now insert (32), (33) into (18); after carrying out the m -integration and simplifying one sees that both terms give the same contribution. Dividing by h , the linear response function is therefore

$$R(t, t_w) = 2 \int dE \Delta^2(E) e^{-\Gamma_0(E)(t-t_w)} \rho(E) P_0(E, t_w). \quad (34)$$

Although I had implicitly assumed an exponential $\rho(E)$ above, this result obviously remains valid also for other $\rho(E)$, as long as T is much smaller than the energy scale over which $\rho(E)$ and $\rho(m|E)$ vary significantly. Comparing with (13), one now sees that in general there is no simple relation between the response and correlation functions for the Barrat and Mézard model. However, for the exponential trap distribution $\rho(E) = T_g^{-1} \exp(E/T_g)$ such a relation does exist. For low T one can approximate the transition rates by a step function, $w_0(E' \leftarrow E) \approx \Theta(E - E')$ and the total exit rates are

$$\Gamma_0(E) = \int_{-\infty}^E dE' \rho(E') = e^{E/T_g} = T_g \rho(E). \quad (35)$$

Thus, comparing (13) and (34) gives

$$R(t, t_w) = -\frac{2}{T_g} \frac{\partial C}{\partial t}. \quad (36)$$

Surprisingly, this is not dissimilar to the result (26) which one obtains for Bouchaud's model in the most natural case $\zeta = 1$: the only difference is in the prefactor, which is $1/T$ for Bouchaud's model but $2/T_g$ for the Barrat and Mézard model considered here.

A simple application of (36) is to the case of a neutral observable, with $\Delta^2(E) = 1$ (say) independently of E . Then from (13) one sees that $C(t, t_w)$ is the hopping correlation function, i.e. the probability of not leaving the current trap between t_w and t . This was worked out by Barrat and Mézard [2] for $T \rightarrow 0$, with the result that $C(t, t_w) = t_w/t$ for long times. Equation (36) then yields $R(t, t_w) = (2/T_g)t_w/t^2$; the step response follows as

$$\chi(t, t_w) = \int_{t_w}^t dt' R(t, t') = \frac{1}{T_g} \left[1 - \left(\frac{t_w}{t} \right)^2 \right] = \frac{1}{T_g} [1 - C^2(t, t_w)]. \quad (37)$$

An FD plot of χ versus C therefore has a parabolic shape, with vanishing asymptotic slope $\partial\chi/\partial C$ for $C \rightarrow 0$, i.e. $X_\infty = 0$. The above calculation shows that the result (37) is exact for the Barrat and Mézard model in the limit $T \rightarrow 0$; it is also consistent with simulation results as shown in figure 2.

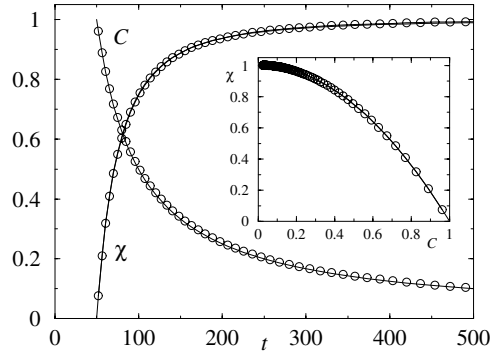


Figure 2. Correlation and response of neutral observables, for the Barrat and Mézard model at $T = 0$; energies are scaled such that $T_g = 1$. Main plot: $C(t, t_w)$ and $\chi(t, t_w)$ against t , for $t_w = 50$. The response was determined for field $h = 0.1$, which can be checked to be in the linear regime. Circles: simulation results (see the appendix). Lines: theoretical predictions $C = t_w/t$ [2] and $\chi = 1 - (t_w/t)^2$, see equation (37). Inset: FD plot of χ versus C .

We can now assess the accuracy of the approximation of multiplicatively perturbed rates (28). From (29), one finds in this case [10], by arguments analogous to those given above, that $\chi = \gamma(1 - C) + \frac{\mu}{2}(1 - C^2)$. Recalling that $\gamma = \mu = 1/T_g$, this is seen to be rather different from (37). In particular, the approximation of multiplicatively perturbed rates incorrectly predicts a non-vanishing asymptotic slope of the FDT plot, $\partial\chi/\partial C = -\gamma = -1/T_g$.

Finally, it is worth discussing a difference between the $T \rightarrow 0$ correlation and response functions for neutral observables, in terms of their dependence of $\rho(E)$. The hopping correlation function is *independent* of $\rho(E)$, as shown in [2]. Within the formalism used here, this is clear if in (13) one sets $\Delta^2(E) = 1$ and changes variables to the cumulative trap density $r(E) = \int_{-\infty}^E dE' \rho(E')$. Together with the first part of (35) this gives $C(t, t_w) = \int_0^1 dr e^{-r(t-t_w)} P_0(r, t_w)$. Since $P_0(r, t_w)$ is independent of $\rho(E)$ (as can be shown from (2) using the same change of variable), the same then holds for $C(t, t_w)$. The intuitive reason for this independence is that the $T \rightarrow 0$ Glauber rates $w_0(E' \leftarrow E) = \Theta(E - E')$ depend only on the relative ‘height’ of departure and arrival traps, but not otherwise on the actual values of E and E' ; correspondingly, the total exit rate $\Gamma_0(E)$ depends only on how many traps are there at energies below E , i.e. on $r(E)$.

By contrast, the response function does depend on $\rho(E)$: transforming from E to r in (34) gives $R(t, t_w) = 2 \int_0^1 dr e^{-r(t-t_w)} \rho(E(r)) P_0(r, t_w)$ and the dependence on $\rho(E)$ cannot be eliminated. This can be explained intuitively by noting that the perturbation term $-hm$ which shifts the energies E introduces an energy scale which is not present for $h = 0$. The response is sensitive to how many traps there are with energies near (measured on this scale) that of the departure trap, and hence to $\rho(E)$.

6. Conclusion

In this paper I have considered mean-field trap models, which are simple and intuitive models of glassy dynamics. I showed briefly that a relation between out-of-equilibrium correlation and response functions in these models, first given by Bouchaud and Dean, is valid for a general class of mean-field trap models; it requires only that the transition rates are affected in the simple multiplicative way (19) by an applied field.

I then considered the Barrat and Mézard model, which has Glauber dynamics and an exponential distribution of trap depths. Glassiness arises in this model from entropic barriers, rather than energetic ones as in the Bouchaud model, and so it is of interest to compare the FD relations that result from these different physical mechanisms. In the natural version of the model where the effect of a field is to shift the energies of all traps according to the usual prescription $E \rightarrow E - hm$, the effect on the transition rates is not simply multiplicative. The out-of-equilibrium response can nevertheless be obtained exactly for low T , and one finds a relation which is quite similar to, but distinct from, that given by Bouchaud and Dean. The exact calculation also shows that an approximate treatment using multiplicatively perturbed rates [10] gives qualitatively incorrect results.

Comparing the above results for the (natural) Barrat and Mézard model with those for Bouchaud's model (with, likewise, the natural choice $\zeta = 1$), one notes two intriguing parallels for the low-temperature out-of-equilibrium dynamics. Firstly, both models give FD plots with $X_\infty = 0$, i.e. with a slope $\partial\chi/\partial C$ which tends to *zero* in the limit $C \rightarrow 0$. (For the Barrat and Mézard model with non-neutral observables³ this can be deduced by applying the arguments of [10] to the relation (36).) Second, the value of the susceptibility itself in the same limit is $\chi_\infty = 1/T_g$ in both models for neutral observables; see again (37). This is precisely the value that one would expect if, as T is lowered, χ_∞ 'freezes' at $T = T_g$ and remains independent of T for $T < T_g$. For Bouchaud's model this T -independence can indeed be shown [10]; for the Barrat and Mézard model the result $\chi_\infty = 1/T_g$ found above for $T \rightarrow 0$ strongly suggests that χ_∞ is likewise T -independent for $0 < T < T_g$. Even though the slow out-of-equilibrium dynamics in the two models is very different, being caused by activation over energy barriers for Bouchaud's model and by entropic barriers for the Barrat and Mézard model, we thus have the intriguing observation that some features of the out-of-equilibrium FD relations are shared. It will be interesting to explore whether this correspondence extends to other properties, and possibly to other models of glassy dynamics.

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Appendix. Simulation method

To simulate any mean-field trap model, one can use that in the limit $N \rightarrow \infty$ no trap is visited twice, so that E and m can be sampled anew at each transition and no explicit population of traps needs to be maintained. The probability for making a transition from a trap with (E, m) to one with (E', m') is

$$P_1(E', m' | E, m) = \Gamma^{-1}(E, m) \rho(m' | E') \rho(E') w(E', m' \leftarrow E, m) \quad (\text{A.1})$$

and contains $\Gamma(E, m)$, the total exit rate from the current trap, as a normalization factor; see (9). Now specialize to the Barrat and Mézard model, with $\rho(E) = T_g^{-1} \exp(E/T_g)$, $E < 0$

³ Strictly speaking, as shown in [10], one requires observables that probe only the ageing dynamics, in the sense that their correlation function only decays on timescales that grow with t_w . A counterexample would be an observable that is sensitive only to the very shallow traps, which in Bouchaud's model gives a correlation function that decays completely on timescales of $\mathcal{O}(1)$ [9].

and a neutral observable for which I take $\rho(m|E) \equiv \rho(m)$ as a zero mean, unit variance Gaussian independently of E . The transition rates at $T = 0$ are $w(E', m' \leftarrow E, m) = \Theta(E - hm - E' + hm')$. Integrating over E' in (A.1) then gives

$$P_1(m'|E, m) = \Gamma^{-1}(E, m)\rho(m')e^{(E-hm+hm')/T_g} \propto \rho(m')e^{hm'}. \quad (\text{A.2})$$

Dividing (A.1) by this yields

$$P_1(E'|m', E, m) = \Theta(E - hm + hm' - E')T_g^{-1}e^{(E'-E+hm-hm')/T_g} \quad (\text{A.3})$$

which is just an exponential distribution over $-\infty < E' < E - hm + hm'$. One can thus sample from (A.1) by first sampling m' from (A.2), which is a Gaussian with unit variance and mean h ; after that one samples E' from (A.3). The total exit rate follows, e.g., from normalization of (A.2) as

$$\Gamma(E, m) = \exp[(E - hm)/T_g + h^2/(2T_g^2)]. \quad (\text{A.4})$$

It is important to note from (A.3) that the distribution of E' depends on $m - m'$. One might be tempted to neglect this dependence, replacing (A.3) by $\Theta(E - E')T_g^{-1}e^{(E'-E)/T_g}$ [10]. However, by repeating the calculations leading to (A.2), (A.3), (A.4) one easily checks that this is equivalent to changing from the exact rates (27) to the multiplicatively perturbed rates (28) with $\gamma = \mu = 1/T_g$. As shown above, this leads to rather different response functions; the precise form of (A.3) is thus important to get the correct results.

The results shown in figure 2 were obtained for a quench from $T = \infty$ at $t = 0$, corresponding to the initial condition $P(E, m, 0) = \rho(m)\rho(E)$, and averaged over 5×10^7 runs. Direct simulations with a population of $N = 10^8$ traps yielded equivalent results, though one needs to be aware of finite- N effects which become more acute for low E because of the exponential decrease in the population density $\rho(E)$.

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